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Surds

Simplifying & Rationalising: GCSE & IGCSE Practice Questions with Worked Solutions

GRADE 9 TARGETED

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13 exam-style questions, 46 marks, grades 5 to 9. Full worked solutions and mark schemes.

Practice Questions

Try each question on paper first. Full worked solutions and mark schemes follow in the next section.

1. Simplify $\sqrt{6}\,(\sqrt{24}+5)-\sqrt{54}$

Give your answer in the form $a+b\sqrt{6}$, where a and b are integers.

GRADE 5

$a =$ .... , $b =$ ....

[Total 3 marks]

2. Write $\dfrac{\sqrt{27}+12}{\sqrt{3}}$ in the form $a+b\sqrt{3}$, where a and b are integers.

GRADE 6

$a =$ .... , $b =$ ....

[Total 3 marks]

3. $(4+\sqrt{5})(3-2\sqrt{5})$ can be written in the form $a+b\sqrt{5}$, where a and b are integers.

Find the value of a and the value of b.

GRADE 7

$a =$ .... , $b =$ ....

[Total 2 marks]

4. Simplify $\sqrt{45}+(\sqrt{5})^3-2\sqrt{20}$

Give your answer in the form $a\sqrt{b}$, where a and b are integers.

GRADE 7

$a =$ .... , $b =$ ....

[Total 3 marks]

5. Simplify $(\sqrt{3})^6 - (\sqrt{3})^5 + (\sqrt{3})^4 - (\sqrt{3})^3$

Give your answer in the form $a + b\sqrt{3}$, where a and b are integers.

GRADE 7

$a =$ .... , $b =$ ....

[Total 3 marks]

6. Write $\dfrac{\sqrt{432}}{\sqrt{2}} + 3\sqrt{54} - \sqrt{384}$ in the form $a\sqrt{b}$, where a and b are integers.

GRADE 7

$a =$ .... , $b =$ ....

[Total 4 marks]

7. Write $\dfrac{35}{\sqrt{7}} - \sqrt{63} - \dfrac{84}{\sqrt{28}}$ in the form $a\sqrt{7}$, where a is an integer.

GRADE 8

$a =$ ....

[Total 4 marks]

8. Express $\dfrac{3+\sqrt{5}}{7-3\sqrt{5}}$ in the form $a + b\sqrt{5}$, where a and b are integers.

GRADE 8

$a =$ .... , $b =$ ....

[Total 4 marks]

9. Expand and simplify $(1 - 2\sqrt{2})^3$

Give your answer in the form $a + b\sqrt{2}$, where a and b are integers.

GRADE 8

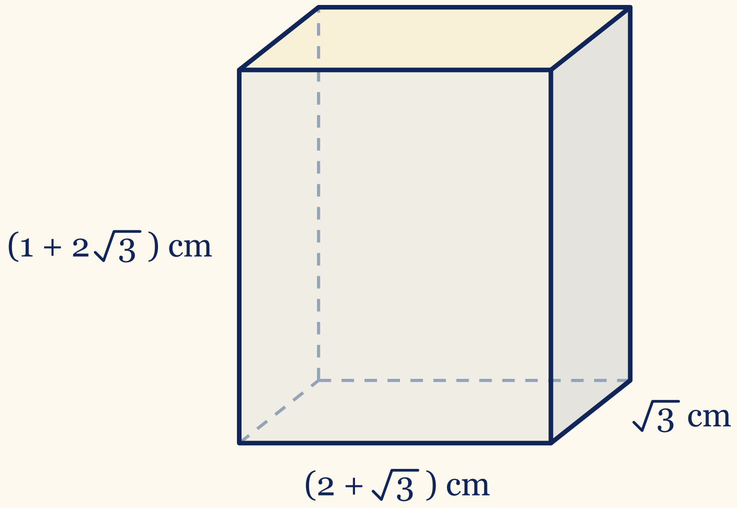
$a =$ .... , $b =$ ....

[Total 3 marks]

10.

Not drawn accurately

GRADE 8



A solid wooden block is in the shape of a cuboid.

Find the exact volume of the block.

Give your answer in the form $a + b\sqrt{3}$, where a and b are integers.

Volume = ... cm^3

[Total 4 marks]

11. Simplify $\dfrac{(3+2\sqrt{6})^2}{3+\sqrt{6}}$

GRADE 8

Give your answer in the form $a + b\sqrt{6}$, where a and b are integers.

$a =$ .... , $b =$ ....

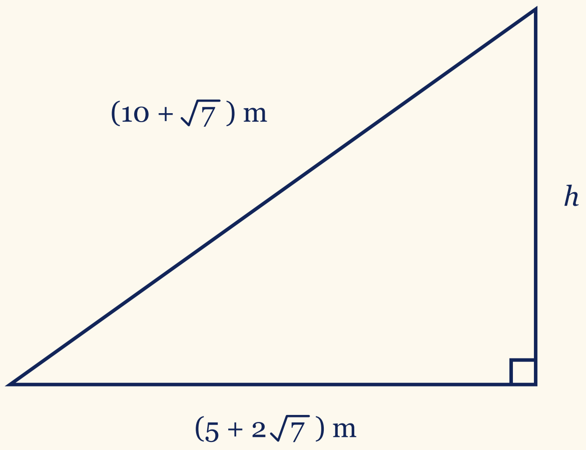
[Total 4 marks]

12.

GRADE 9

Problem solving

Not drawn accurately



A ramp leads up to a raised platform.

The sloping surface of the ramp is $(10 + \sqrt{7})$ metres long.

Along the ground, the ramp starts $(5 + 2\sqrt{7})$ metres from the foot of the platform.

Find the exact height, h, of the platform. Give your answer in its simplest form.

$h =$ .... m

[Total 4 marks]

13. Simplify $\dfrac{7\sqrt{2}}{3+\sqrt{2}} + \dfrac{4+\sqrt{2}}{2-\sqrt{2}}$

GRADE 9

Give your answer in the form $a + b\sqrt{2}$, where a and b are integers.

$a =$ .... , $b =$ ....

[Total 5 marks]

Worked Solutions & Mark Schemes

Every mark (M for method, A for accuracy) is shown, just like a real examiner.

Question 1 - Exam Solution

Understanding the Question

Given

$\sqrt{6}\left(\sqrt{24}+5\right)-\sqrt{54}$

A single surd multiplying a bracket, and one more surd to simplify.

Find

The integers a and b in $a+b\sqrt{6}$. Watch the first product. Two surds multiplied together can collapse into a whole number, and here they do.

Plan the Solution

- Expand the bracket first. The surd outside multiplies every term inside, not just the first.
- Two surds multiplied become one surd: the numbers under the roots multiply together.
- That product is a square number here, so the surd disappears completely.
- Simplify the last surd by pulling out its largest square factor, then collect.

Worked Solution [3 marks]

> **Rule - Multiplying surds:** $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$. The numbers under the roots multiply into one root. If that product is a square number, the surd vanishes and a whole number is left. Then $\sqrt{k^2n}=k\sqrt{n}$, so always pull out the largest square factor.

Step 1: Expand the bracket.

$$\sqrt{6}\left(\sqrt{24}+5\right)=\sqrt{6}\times\sqrt{24}+\sqrt{6}\times 5=\sqrt{6}\times\sqrt{24}+5\sqrt{6}$$

(Reason: the surd outside multiplies every term inside the bracket, not just the first one.)

Step 2: Multiply the two surds.

$$\sqrt{6}\times\sqrt{24}=\sqrt{6\times 24}=\sqrt{144}=12$$

(Reason: the numbers under the roots multiply into one root. One hundred and forty four is a square number, so the surd vanishes completely and a whole number is left. Leaving this as a surd is the mistake that costs the mark.)

Step 3: Simplify the last surd.

$\sqrt{54} = \sqrt{9 \times 6} = 3\sqrt{6}$

(Reason: nine is the largest square factor of fifty four.)

Step 4: Collect the whole numbers and the surds.

$12 + 5\sqrt{6} - 3\sqrt{6} = 12 + 2\sqrt{6}$

(Reason: a whole number and a surd never merge. They are two separate running totals.)

$$a = 12, \quad b = 2$$

Verification

- ✓ **Check 1:** The expression evaluates to 16.8990 and $12 + 2\sqrt{6} \approx 16.8990$, so the two agree numerically.
- ✓ **Check 2:** Simplify $\sqrt{24}$ first instead, and never write $\sqrt{144}$ at all: $\sqrt{24} = 2\sqrt{6}$, so $\sqrt{6} \times 2\sqrt{6} = 2 \times 6 = 12$. Same whole number, a different route.
- ✓ **Check 3:** If $\sqrt{6} \times \sqrt{24}$ had been left as a surd, there would be no whole number in the answer at all, and the form $a + b\sqrt{6}$ could not be reached. The very existence of an integer a proves that first product collapsed. Also $6 = 2 \times 3$ has no repeated prime, so it carries no square factor: the answer is fully simplified.

Mark Scheme Breakdown

Step	Mark	Description	Got it?
Multiply the two surds: root 6 times root 24 = root 144 = 12, by any valid route	M1	Method - multiplying surds	✓
Simplify root 54 = 3 root 6	M1	Method - largest square factor	✓
Give a = 12 and b = 2	A1	Accuracy - final answer	✓

Full marks: 3/3

Question 2 - Exam Solution

Understanding the Question

Given

Find

$$\frac{\sqrt{27}+12}{\sqrt{3}}$$

One fraction, with a single surd on the bottom and two terms on the top.

The integers a and b in $a + b\sqrt{3}$. Both terms on the top must be dealt with, not just the first one. That is the trap.

Plan the Solution

- A surd on the bottom of a fraction is never an acceptable answer. Clear it first.
- The bottom is a single surd, so multiplying the top and the bottom by that same surd is enough. No conjugate is needed here.
- The top has two terms. Every one of them must be multiplied.
- Then divide every term on the top by the whole number left on the bottom.

Worked Solution [3 marks]

> **Rule - Rationalising a single-term denominator:** when the bottom is a single surd, multiply the top and the bottom by that surd. Since $\sqrt{n} \times \sqrt{n} = n$, the bottom becomes a whole number. Every term on the top must be multiplied, and then every term must be divided.

Step 1: Multiply the top and the bottom by the surd.

$$\frac{\sqrt{27}+12}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{(\sqrt{27}+12)\sqrt{3}}{(\sqrt{3})^2}$$

(Reason: multiplying by root three over root three is multiplying by one, so the value does not change. Only its form does. The bottom is now root three squared, which is three, a whole number.)

Step 2: Multiply out the top.

$$(\sqrt{27}+12)\sqrt{3} = \sqrt{27} \times \sqrt{3} + 12\sqrt{3} = \sqrt{81} + 12\sqrt{3} = 9 + 12\sqrt{3}$$

(Reason: both terms on the top are multiplied, not just the first. Twenty seven times three is eighty one, a square number, so that surd disappears completely and a whole number is left.)

Step 3: Divide every term by the whole number on the bottom.

$$\frac{9+12\sqrt{3}}{3} = \frac{9}{3} + \frac{12\sqrt{3}}{3} = 3 + 4\sqrt{3}$$

(Reason: every term on the top is divided, not just the first one. This is where the marks are most often thrown away.)

$$a = 3, \quad b = 4$$

Verification

✓ **Check 1:** The expression evaluates to 9.9282 and $3 + 4\sqrt{3} \approx 9.9282$, so the two agree numerically.

✓ **Check 2:** Split the fraction in two instead: $\dfrac{\sqrt{27}}{\sqrt{3}} + \dfrac{12}{\sqrt{3}} = \sqrt{9} + \dfrac{12\sqrt{3}}{3} = 3 + 4\sqrt{3}$. The first part is a surd divided by a surd, so it needs no rationalising at all. Same answer, a completely different route.

✓ **Check 3:** Multiply the answer back by the denominator: $(3 + 4\sqrt{3})\sqrt{3} = 3\sqrt{3} + 4 \times 3 = 12 + 3\sqrt{3}$. The original numerator is $\sqrt{27} + 12 = 3\sqrt{3} + 12$, which is the same thing. The division was correct. Also 3 is prime, so it carries no square factor: the answer is fully simplified.

Mark Scheme Breakdown

Step	Mark	Description	Got it?
Clear the surd from the denominator by any valid route, e.g. multiply the top and the bottom by root 3	**M1**	Method - rationalising	✓
Obtain (9 + 12 root 3) over 3, or equivalent	**M1**	Method - both terms on the top	✓
Give a = 3 and b = 4	**A1**	Accuracy - final answer	✓

Full marks: 3/3

Question 3 - Exam Solution

Understanding the Question

Given

$(4 + \sqrt{5})(3 - 2\sqrt{5})$

A product of two surd brackets.

Find

The integers a and b in $a + b\sqrt{5}$.

Plan the Solution

- Expand all four products. Not three.
- Use $\sqrt{5} \times \sqrt{5} = 5$ to turn the surd product into a whole number.
- Collect the whole numbers together and the surd terms together.

Worked Solution [2 marks]

> **Rule - Expanding surd brackets:** multiply every term in the first bracket by every term in the second, then use $\sqrt{n} \times \sqrt{n} = n$ to remove the surd from any $\sqrt{n} \times \sqrt{n}$ product.

Step 1: Expand all four products.

$$(4)(3) + (4)(-2\sqrt{5}) + (\sqrt{5})(3) + (\sqrt{5})(-2\sqrt{5})$$
$$= 12 - 8\sqrt{5} + 3\sqrt{5} - 2(\sqrt{5})^2$$

(Reason: every term in the first bracket meets every term in the second, so four terms come out.)

Step 2: Turn the surd product into a whole number.

$$(\sqrt{5})^2 = 5 \implies -2(\sqrt{5})^2 = -10$$
$$= 12 - 8\sqrt{5} + 3\sqrt{5} - 10$$

(Reason: the square root of 5 is the number that squares to 5. This is the step that turns a into an integer.)

Step 3: Collect like terms.

$$12 - 10 = 2$$
$$-8\sqrt{5} + 3\sqrt{5} = -5\sqrt{5}$$

(Reason: a surd behaves like an algebraic term. Its coefficients add and subtract, but it never merges with a whole number.)

$$a = 2, \quad b = -5$$

Verification

- ✓ **Check 1:** $(4 + \sqrt{5})(3 - 2\sqrt{5}) \approx -9.1803$ and $2 - 5\sqrt{5} \approx -9.1803$, so the two agree numerically.
- ✓ **Check 2:** Let $x = \sqrt{5}$. Then $(4 + x)(3 - 2x) = -2x^2 - 5x + 12$, and with $x^2 = 5$ this gives $2 - 5\sqrt{5}$. Same answer by a different route.
- ✓ **Check 3:** The whole numbers give $12 - 10 = 2$ and the surd coefficients give $-8 + 3 = -5$. Both parts agree.

Mark Scheme Breakdown

Step	Mark	Description	Got it?
Expand the brackets, at least 3 of the 4 terms correct	M1	Method - correct expansion	✓

Step	Mark	Description	Got it?
Simplify to give a = 2 and b = -5	A1	Accuracy - both values	✓

Full marks: 2/2

Question 4 - Exam Solution

Understanding the Question

Given

$\sqrt{45} + (\sqrt{5})^3 - 2\sqrt{20}$

Three surd terms, none of them in simplest form.

Find

The integers a and b in $a\sqrt{b}$.

Plan the Solution

- Simplify each term on its own first. Nothing can be added until all three share the same surd.
- $\sqrt{45}$ and $2\sqrt{20}$ each hide a square factor. Pull it out.
- $(\sqrt{5})^3$ is a power, not a simplification. Split it as $(\sqrt{5})^2 \times \sqrt{5}$.
- Only then collect the coefficients.

Worked Solution [3 marks]

> **Rule - Simplifying surds:** $\sqrt{k^2 n} = k\sqrt{n}$, so pull out the largest square factor. A surd cubed follows from the same idea: $(\sqrt{n})^3 = (\sqrt{n})^2 \times \sqrt{n} = n\sqrt{n}$. Only surds with the same number under the root can be added or subtracted.

Step 1: Simplify the first surd.

$$\sqrt{45} = \sqrt{9 \times 5} = \sqrt{9} \times \sqrt{5} = 3\sqrt{5}$$

(Reason: nine is the largest square factor of forty-five. Its square root, three, comes outside the root.)

Step 2: Evaluate the cube.

$$(\sqrt{5})^3 = (\sqrt{5})^2 \times \sqrt{5} = 5 \times \sqrt{5} = 5\sqrt{5}$$

(Reason: a square root squared gives the number back. The third root is left outside.)

Step 3: Simplify the third surd.

$$2\sqrt{20} = 2\sqrt{4 \times 5} = 2 \times 2\sqrt{5} = 4\sqrt{5}$$

(Reason: the two in front stays. It multiplies whatever comes out of the root.)

Step 4: Collect the like surds.

$3\sqrt{5} + 5\sqrt{5} - 4\sqrt{5} = (3 + 5 - 4)\sqrt{5} = 4\sqrt{5}$

(Reason: all three terms are now multiples of the same surd, so only the coefficients are added.)

$$a = 4, \quad b = 5$$

Verification

✓ **Check 1:** $\sqrt{45} + (\sqrt{5})^3 - 2\sqrt{20} \approx 8.9443$ and $4\sqrt{5} \approx 8.9443$, so the two agree numerically.

✓ **Check 2:** Squaring the answer gives $(4\sqrt{5})^2 = 16 \times 5 = 80$, and squaring the original expression also gives exactly 80. A different operation, the same agreement.

✓ **Check 3:** $b = 5$ is prime, so it has no square factor left in it. The answer cannot be reduced any further, which is what the form $a\sqrt{b}$ requires.

Mark Scheme Breakdown

Step	Mark	Description	Got it?
Correctly simplify at least one surd term, e.g. root 45 = 3 root 5 or 2 root 20 = 4 root 5	M1	Method - simplifying a surd	✓
Correctly evaluate the cube, (root 5) cubed = 5 root 5	M1	Method - evaluating the power	✓
Give a = 4 and b = 5	A1	Accuracy - final answer	✓

Full marks: 3/3

Question 5 - Exam Solution

Understanding the Question

Given

$(\sqrt{3})^6 - (\sqrt{3})^5 + (\sqrt{3})^4 - (\sqrt{3})^3$

Find

The integers a and b in $a + b\sqrt{3}$.

Everything here follows from one fact:

Four powers of the same surd, with alternating signs.

$(\sqrt{3})^2 = 3$.

Plan the Solution

- One fact does all the work: $(\sqrt{3})^2 = 3$.
- An even power of a surd is always a whole number. Build it by squaring in pairs.
- An odd power is always a whole number times the surd. It is the even power below it, multiplied by one more root.
- Substitute the four values back in, then collect the whole numbers and the surds separately.

Worked Solution [3 marks]

> **Rule - Powers of a surd:** everything follows from $(\sqrt{n})^2 = n$. An even power is a whole number, because the roots pair up. An odd power is a whole number times $\sqrt{n}$, because one root is left over with no partner.

Step 1: The even powers are whole numbers.

$$(\sqrt{3})^4 = \left((\sqrt{3})^2\right)^2 = 3^2 = 9$$
$$(\sqrt{3})^6 = \left((\sqrt{3})^2\right)^3 = 3^3 = 27$$

(Reason: the roots pair off. Six roots make three pairs, and each pair is a three.)

Step 2: The odd powers keep one surd.

$$(\sqrt{3})^3 = (\sqrt{3})^2 \times \sqrt{3} = 3\sqrt{3}$$
$$(\sqrt{3})^5 = (\sqrt{3})^4 \times \sqrt{3} = 9\sqrt{3}$$

(Reason: an odd number of roots always leaves one without a partner, so a single root stays outside.)

Step 3: Substitute the four values back in.

$$27 - 9\sqrt{3} + 9 - 3\sqrt{3}$$

(Reason: keep the signs from the original expression. It is the minus signs that most students lose here.)

Step 4: Collect the whole numbers and the surds separately.

$$27 + 9 = 36$$
$$-9\sqrt{3} - 3\sqrt{3} = -12\sqrt{3}$$

(Reason: a whole number and a surd never merge. They are two separate running totals.)

$$a = 36, \quad b = -12$$

Verification

✓ **Check 1:** The expression evaluates to 15.2154 and $36 - 12\sqrt{3} \approx 15.2154$, so the two agree numerically.

✓ **Check 2:** Take out $(\sqrt{3})^3 = 3\sqrt{3}$ as a common factor: $3\sqrt{3}\left[3\sqrt{3} - 3 + \sqrt{3} - 1\right] = 3\sqrt{3}\left[4\sqrt{3} - 4\right] = 36 - 12\sqrt{3}$. Same answer, a completely different route.

✓ **Check 3:** The even powers, the sixth and the fourth, produced whole numbers and both carried a plus sign, so a must be positive. The odd powers, the fifth and the third, produced surds and both carried a minus sign, so b must be negative. The signs of the answer are forced by the structure, and they agree.

Mark Scheme Breakdown

Step	Mark	Description	Got it?
Correctly evaluate at least two of the four powers	**M1**	Method - powers of a surd	✓
Correctly evaluate all four powers	**M1**	Method - all four correct	✓
Give a = 36 and b = -12	**A1**	Accuracy - final answer	✓

Full marks: 3/3

Question 6 - Exam Solution

Understanding the Question

Given

$$\frac{\sqrt{432}}{\sqrt{2}} + 3\sqrt{54} - \sqrt{384}$$

The first term is a surd divided by a surd, not a number over a surd.

Find

The integers a and b in $a\sqrt{b}$. Nothing can be combined until all three terms are multiples of the same surd.

Plan the Solution

- The first term is not a rationalising problem. Two surds divided become one surd.
- $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$. Do the division under one root and the numbers get much smaller.
- Simplify the other two terms by pulling out their largest square factors.

- Only then collect the coefficients.

Worked Solution [4 marks]

Rule - Dividing surds: $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$. Two surds divided become one surd. This is not the same as rationalising, which is what you do when the top is a whole number. Here both parts are surds, so they simply merge.

Step 1: Divide the two surds under one root.

$$\frac{\sqrt{432}}{\sqrt{2}} = \sqrt{\frac{432}{2}} = \sqrt{216} = \sqrt{36 \times 6} = 6\sqrt{6}$$

(Reason: dividing four hundred and thirty two by two first turns an awkward number into a manageable one. Rationalising instead would give the square root of eight hundred and sixty four, which is worse.)

Step 2: Simplify the coefficient surd.

$$3\sqrt{54} = 3\sqrt{9 \times 6} = 3 \times 3\sqrt{6} = 9\sqrt{6}$$

(Reason: the three in front stays. It multiplies whatever comes out of the root.)

Step 3: Simplify the plain surd.

$$\sqrt{384} = \sqrt{64 \times 6} = 8\sqrt{6}$$

(Reason: sixty four is the largest square factor of three hundred and eighty four. Missing it and using four instead would leave two root ninety six, which is not fully simplified.)

Step 4: Collect the like surds.

$$6\sqrt{6} + 9\sqrt{6} - 8\sqrt{6} = (6 + 9 - 8)\sqrt{6} = 7\sqrt{6}$$

(Reason: all three are now multiples of the same surd, so only the coefficients are combined.)

$$a = 7, \quad b = 6$$

Verification

✓ **Check 1:** The expression evaluates to 17.1464 and $7\sqrt{6} \approx 17.1464$, so the two agree numerically.

✓ **Check 2:** Rationalise the first term instead of dividing: $\dfrac{\sqrt{432}}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{864}}{2} = \dfrac{12\sqrt{6}}{2} = 6\sqrt{6}$. Same term, different method.

✓ **Check 3:** $b = 6 = 2 \times 3$, with no repeated prime factor, so six has no square factor. The answer cannot be reduced any further.

Mark Scheme Breakdown

Step	Mark	Description	Got it?
Obtain root 432 over root 2 = 6 root 6, by any valid route	**M1**	Method - dividing surds	✓
Simplify 3 root 54 = 9 root 6	**M1**	Method - coefficient surd	✓
Simplify root 384 = 8 root 6	**M1**	Method - largest square factor	✓
Give a = 7 and b = 6	**A1**	Accuracy - final answer	✓

Full marks: 4/4

Question 7 - Exam Solution

Understanding the Question

Given

$$\frac{35}{\sqrt{7}} - \sqrt{63} - \frac{84}{\sqrt{28}}$$

Two of the three terms have a surd on the bottom.

Find

The integer a in $a\sqrt{7}$. Note that a may be negative. The question says integer, not positive integer.

Plan the Solution

- A surd on the bottom of a fraction is never an acceptable answer. Clear it first.
- Deal with each term separately until all three are a multiple of the same surd.
- $\sqrt{28}$ can be simplified before rationalising. Doing that first keeps the numbers small.
- Only then collect the coefficients.

Worked Solution [4 marks]

Rule - Rationalising a denominator: a surd must never be left on the bottom of a fraction. Multiply the top and the bottom by that surd. Since $\sqrt{n} \times \sqrt{n} = n$, the bottom becomes a whole number. If the denominator can be simplified first, do that, because it keeps the numbers small.

Step 1: Rationalise the first fraction.

$$\frac{35}{\sqrt{7}} = \frac{35}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{35\sqrt{7}}{7} = 5\sqrt{7}$$

(Reason: multiplying by root seven over root seven is multiplying by one, so the value does not change. Only its form does.)

Step 2: Simplify the plain surd.

$$\sqrt{63} = \sqrt{9 \times 7} = 3\sqrt{7}$$

(Reason: nine is the largest square factor of sixty-three.)

Step 3: Simplify the denominator, then rationalise.

$$\sqrt{28} = \sqrt{4 \times 7} = 2\sqrt{7}$$

$$\frac{84}{\sqrt{28}} = \frac{84}{2\sqrt{7}} = \frac{42}{\sqrt{7}} = \frac{42\sqrt{7}}{7} = 6\sqrt{7}$$

(Reason: simplifying the bottom first turns eighty-four into forty-two. Rationalising directly also works, but the numbers are larger.)

Step 4: Collect the like surds.

$$5\sqrt{7} - 3\sqrt{7} - 6\sqrt{7} = (5 - 3 - 6)\sqrt{7} = -4\sqrt{7}$$

(Reason: all three are multiples of the same surd, so only the coefficients are combined. The answer is negative, and that is allowed.)

$$a = -4$$

Verification

✓ **Check 1:** The expression evaluates to -10.5830 and $-4\sqrt{7} \approx -10.5830$, so the two agree numerically.

✓ **Check 2:** Rationalise the last term without simplifying first: $\frac{84}{\sqrt{28}} \times \frac{\sqrt{28}}{\sqrt{28}} = \frac{84\sqrt{28}}{28} = 3\sqrt{28} = 3 \times 2\sqrt{7} = 6\sqrt{7}$. Same term, different method.

✓ **Check 3:** $\frac{84}{\sqrt{28}} \approx 15.87$, which is larger than $\frac{35}{\sqrt{7}} - \sqrt{63} \approx 5.29$. Subtracting it must leave a negative result, so a positive answer here would be wrong on sight.

Mark Scheme Breakdown

Step	Mark	Description	Got it?
Rationalise the first fraction, 35 over root 7 = 5 root 7	**M1**	Method - rationalising	✓
Simplify the plain surd, root 63 = 3 root 7	**M1**	Method - simplifying a surd	✓
Obtain 84 over root 28 = 6 root 7, by any valid route	**M1**	Method - simplify then rationalise	✓
Give a = -4	**A1**	Accuracy - final answer	✓

Full marks: 4/4

Question 8 - Exam Solution

Understanding the Question

Given

$$\frac{3+\sqrt{5}}{7-3\sqrt{5}}$$

The denominator has two terms, one of them a surd.

Find

The integers a and b in $a+b\sqrt{5}$. Multiplying by $\sqrt{5}$ alone will not clear this denominator. That is the trap.

Plan the Solution

- The denominator is $7-3\sqrt{5}$. Its conjugate is $7+3\sqrt{5}$: the same two terms with the sign between them flipped.
- Multiply the top and the bottom by that conjugate.
- The bottom becomes a difference of two squares, and the surd vanishes.
- Expand the top carefully. All four products. Then divide every term by the whole number left on the bottom.

Worked Solution [4 marks]

> **Rule - Rationalising with a conjugate:** when the denominator has two terms and one is a surd, multiply the top and the bottom by the conjugate, which is the same expression with the middle sign flipped. Then $(c-d\sqrt{n})(c+d\sqrt{n})=c^2-d^2n$, a whole number, and the surd is gone.

Step 1: Multiply top and bottom by the conjugate.

$$\frac{3+\sqrt{5}}{7-3\sqrt{5}} = \frac{(3+\sqrt{5})(7+3\sqrt{5})}{(7-3\sqrt{5})(7+3\sqrt{5})}$$

(Reason: the conjugate is the same expression with the middle sign flipped. Doing it to the top and the bottom is multiplying by one, so the value does not change.)

Step 2: Expand the numerator.

$$(3)(7)+(3)(3\sqrt{5})+(\sqrt{5})(7)+(\sqrt{5})(3\sqrt{5})$$
$$= 21+9\sqrt{5}+7\sqrt{5}+15 = 36+16\sqrt{5}$$

(Reason: all four products. The last one is three times root five squared, which is fifteen, and that is where the whole number comes from.)

Step 3: Expand the denominator.

$$(7-3\sqrt{5})(7+3\sqrt{5}) = 7^2-(3\sqrt{5})^2 = 49-9\times 5 = 49-45 = 4$$

(Reason: difference of two squares. The middle terms cancel, so no expanding is needed. Note that three root five, squared, is nine times five, not three times five.)

Step 4: Divide through.

$$\frac{36+16\sqrt{5}}{4} = \frac{36}{4}+\frac{16\sqrt{5}}{4} = 9+4\sqrt{5}$$

(Reason: every term on the top is divided by the whole number on the bottom.)

$$a = 9, \quad b = 4$$

Verification

✓ **Check 1:** The fraction evaluates to 17.9443 and $9+4\sqrt{5}\approx 17.9443$, so the two agree numerically.

✓ **Check 2:** Multiply the answer back by the original denominator: $(9+4\sqrt{5})(7-3\sqrt{5}) = 63-27\sqrt{5}+28\sqrt{5}-60 = 3+\sqrt{5}$, which is exactly the original numerator. The division was correct.

✓ **Check 3:** The denominator $7-3\sqrt{5}\approx 0.29$ is tiny, while the numerator is about 5.24. Dividing by something that small must give a large answer, and $9+4\sqrt{5}\approx 17.94$ is large. An answer near one would be wrong on sight.

Mark Scheme Breakdown

Step	Mark	Description	Got it?
Multiply the numerator and denominator by the conjugate, 7 + 3 root 5	**M1**	Method - using the conjugate	✓
Expand the numerator to 36 + 16 root 5	**M1**	Method - expanding the numerator	✓
Expand the denominator to 4, using 7 squared minus (3 root 5) squared	**M1**	Method - difference of two squares	✓
Give a = 9 and b = 4	**A1**	Accuracy - final answer	✓

Full marks: 4/4

Question 9 - Exam Solution

Understanding the Question

Given

$(1-2\sqrt{2})^3$

A bracket raised to the third power.

Find

The integers a and b in $a+b\sqrt{2}$. Note that $1-2\sqrt{2}\approx -1.83$, which is negative, so its cube must be negative too.

Plan the Solution

- A cube is not distributive. $(p+q)^3$ is not p^3+q^3. That mistake loses every mark.
- Square the bracket first. That gives a two-term expression.
- Then multiply that result by the original bracket one more time.
- Watch the coefficient: $(2\sqrt{2})^2 = 4\times 2 = 8$, not two times two. The 2 squares as well as the surd.

Worked Solution [3 marks]

> **Rule - Cubing a surd bracket:** $(p+q\sqrt{n})^3 = (p+q\sqrt{n})^2 \times (p+q\sqrt{n})$. Square it once, then multiply by the bracket again. Never cube the terms separately. And remember $(q\sqrt{n})^2 = q^2 n$: the coefficient is squared too.

Step 1: Square the bracket.

$$(1-2\sqrt{2})^2 = 1-4\sqrt{2}+(2\sqrt{2})^2 = 1-4\sqrt{2}+8 = 9-4\sqrt{2}$$

(Reason: two root two, squared, is four times two, which is eight. Squaring only the surd and leaving the two alone is the most common error here.)

Step 2: Multiply that result by the bracket again.

$(9 - 4\sqrt{2})(1 - 2\sqrt{2})$
$= 9 - 18\sqrt{2} - 4\sqrt{2} + 8 \times 2$

(Reason: all four products. The last one is minus four root two times minus two root two, which is plus sixteen. Two negatives make a positive, and the surds multiply to give two.)

Step 3: Collect the whole numbers and the surds.

$9 + 16 = 25$
$-18\sqrt{2} - 4\sqrt{2} = -22\sqrt{2}$

(Reason: a whole number and a surd never merge. They are two separate running totals.)

$$a = 25, \quad b = -22$$

Verification

✓ **Check 1:** $(1 - 2\sqrt{2})^3 \approx -6.1127$ and $25 - 22\sqrt{2} \approx -6.1127$, so the two agree numerically.

✓ **Check 2:** Use the binomial expansion instead, $(p+q)^3 = p^3 + 3p^2q + 3pq^2 + q^3$, with $p = 1$ and $q = -2\sqrt{2}$: $1 - 6\sqrt{2} + 24 - 16\sqrt{2} = 25 - 22\sqrt{2}$. Same answer, a completely different method.

✓ **Check 3:** The bracket $1 - 2\sqrt{2} \approx -1.83$ is negative, and a negative number cubed stays negative. $25 - 22\sqrt{2} \approx -6.11$, which is negative. A positive answer here would be wrong on sight.

Mark Scheme Breakdown

Step	Mark	Description	Got it?
Square the bracket correctly: (1 - 2 root 2) squared = 9 - 4 root 2	M1	Method - squaring the bracket	✓
Multiply that result by (1 - 2 root 2) and expand all four products	M1	Method - the second expansion	✓
Give a = 25 and b = -22	A1	Accuracy - final answer	✓

Full marks: 3/3

Question 10 - Exam Solution

Understanding the Question

Given

Width $= 2 + \sqrt{3}$, height $= 1 + 2\sqrt{3}$, depth $= \sqrt{3}$, all in cm.

Three dimensions, all of them containing a surd.

Find

The volume, in the form $a + b\sqrt{3}$. Exact means no decimals, so the answer stays as a surd.

Plan the Solution

- Volume of a cuboid is width times height times depth. Three things multiplied together.
- Multiply two at a time. Start with the plain surd, because it simplifies immediately.
- $\sqrt{3} \times \sqrt{3} = 3$, so a whole number appears at once and the numbers stay small.
- Then multiply that result by the remaining bracket, and collect.

Worked Solution [4 marks]

> **Rule - Volume with surd sides:** $V = \text{width} \times \text{height} \times \text{depth}$. Multiply two at a time. Choosing the plain surd first is not required, but it keeps the numbers small, because $\sqrt{n} \times \sqrt{n} = n$ straight away.

Step 1: Write down the volume.

$$V = (2 + \sqrt{3}) \times (1 + 2\sqrt{3}) \times \sqrt{3}$$

(Reason: the volume of a cuboid is the product of its three dimensions. The order does not matter.)

Step 2: Multiply the plain surd by the width bracket.

$$\sqrt{3}\,(2 + \sqrt{3}) = 2\sqrt{3} + (\sqrt{3})^2 = 2\sqrt{3} + 3 = 3 + 2\sqrt{3}$$

(Reason: root three times root three is three. A whole number appears immediately, which keeps the next step small.)

Step 3: Multiply that result by the height bracket.

$$(3 + 2\sqrt{3})(1 + 2\sqrt{3})$$
$$= 3 + 6\sqrt{3} + 2\sqrt{3} + 4 \times 3$$

(Reason: all four products. The last one is two root three times two root three, which is four times three, or twelve. Both the coefficient and the surd are multiplied.)

Step 4: Collect the whole numbers and the surds.

$$3 + 12 = 15$$

$6\sqrt{3} + 2\sqrt{3} = 8\sqrt{3}$

(Reason: two separate running totals. A whole number and a surd never merge.)

$$V = 15 + 8\sqrt{3}\text{ cm}^3$$

Verification

✓ **Check 1:** $(2+\sqrt{3})(1+2\sqrt{3})\sqrt{3} \approx 28.8564$ and $15 + 8\sqrt{3} \approx 28.8564$, so the two agree numerically.

✓ **Check 2:** Multiply the two brackets first instead: $(2+\sqrt{3})(1+2\sqrt{3}) = 8 + 5\sqrt{3}$, then $\times\sqrt{3} = 8\sqrt{3} + 5 \times 3 = 15 + 8\sqrt{3}$. Same answer, so the order genuinely does not matter.

✓ **Check 3:** The block is roughly $3.7 \times 4.5 \times 1.7$ cm, so the volume must be around 29 cubic centimetres. $15 + 8\sqrt{3} \approx 28.9$. Consistent.

Mark Scheme Breakdown

Step	Mark	Description	Got it?
Multiply the plain surd by either bracket, e.g. root 3 (2 + root 3) = 3 + 2 root 3	**M1**	Method - multiplying by the surd	✓
Multiply that result by the remaining bracket	**M1**	Method - the second product	✓
Expand all four products correctly	**M1**	Method - correct expansion	✓
Give 15 + 8 root 3, so a = 15 and b = 8	**A1**	Accuracy - final answer	✓

Full marks: 4/4

Question 11 - Exam Solution

Understanding the Question

Given

$$\frac{(3+2\sqrt{6})^2}{3+\sqrt{6}}$$

Find

The integers a and b in $a + b\sqrt{6}$. Two separate techniques are needed here, and

A squared bracket on the top, and a two-term denominator containing a surd.

they must be done in the right order: the square is expanded first, and only then is the denominator cleared.

Plan the Solution

- Expand the square in full. A squared bracket has three terms, not two.
- Nothing can be done about the denominator until the top is a single two-term expression.
- The denominator has two terms, so it needs its conjugate: the same two terms with the middle sign flipped.
- Multiply the top and the bottom by that conjugate, then divide every term on the top by the whole number left on the bottom.

Worked Solution [4 marks]

> **Rule - A squared bracket over a surd denominator:** expand the square first, using $(p+q\sqrt{n})^2 = p^2 + 2pq\sqrt{n} + q^2n$. The coefficient is squared too, so $(q\sqrt{n})^2 = q^2n$. Then rationalise: multiply the top and the bottom by the conjugate, and $(c+d\sqrt{n})(c-d\sqrt{n}) = c^2 - d^2n$, a whole number.

Step 1: Expand the square in full.

$$(3+2\sqrt{6})^2 = 3^2 + 2\times 3\times 2\sqrt{6} + (2\sqrt{6})^2$$
$$= 9 + 12\sqrt{6} + 24 = 33 + 12\sqrt{6}$$

(Reason: three terms, not two. Two root six, squared, is four times six, which is twenty four, not twelve. Squaring only the surd and leaving the two alone is the most common error here, and dropping the middle term loses the surd altogether.)

Step 2: Multiply the top and the bottom by the conjugate.

$$\frac{33+12\sqrt{6}}{3+\sqrt{6}} \times \frac{3-\sqrt{6}}{3-\sqrt{6}} = \frac{(33+12\sqrt{6})(3-\sqrt{6})}{(3+\sqrt{6})(3-\sqrt{6})}$$
$$= \frac{99 - 33\sqrt{6} + 36\sqrt{6} - 72}{9-6} = \frac{27+3\sqrt{6}}{3}$$

(Reason: the denominator has a plus in the middle, so its conjugate has a minus. The bottom is a difference of two squares and becomes three, a whole number. On the top, twelve root six times minus root six is minus twelve times six, which is minus seventy two.)

Step 3: Divide every term by the whole number on the bottom.

$$\frac{27+3\sqrt{6}}{3} = \frac{27}{3} + \frac{3\sqrt{6}}{3} = 9 + \sqrt{6}$$

(Reason: every term on the top is divided, not just the first one. The surd term still has a coefficient of one, so here b = 1.)

$$a = 9, \quad b = 1$$

Verification

✓ **Check 1:** The expression evaluates to 11.4495 and $9 + \sqrt{6} \approx 11.4495$, so the two agree numerically.

✓ **Check 2:** Do not square the bracket at all. Rationalise one factor of it first: $\dfrac{3+2\sqrt{6}}{3+\sqrt{6}} = \dfrac{(3+2\sqrt{6})(3-\sqrt{6})}{3} = \dfrac{-3+3\sqrt{6}}{3} = -1+\sqrt{6}$, then multiply by the remaining bracket: $(-1+\sqrt{6})(3+2\sqrt{6}) = -3 - 2\sqrt{6} + 3\sqrt{6} + 12 = 9 + \sqrt{6}$. Same answer, a completely different route.

✓ **Check 3:** Multiply the answer back by the denominator: $(9+\sqrt{6})(3+\sqrt{6}) = 27 + 9\sqrt{6} + 3\sqrt{6} + 6 = 33 + 12\sqrt{6}$, which is exactly the expanded numerator. The division was correct. Also, $6 = 2 \times 3$ has no repeated prime, so it carries no square factor: the form $a + b\sqrt{6}$ is fully simplified.

Mark Scheme Breakdown

Step	Mark	Description	Got it?
Expand the square in full: (3 + 2 root 6) squared = 33 + 12 root 6	M1	Method - squaring the bracket	✓
Multiply the top and the bottom by the conjugate 3 - root 6	M1	Method - using the conjugate	✓
Obtain (27 + 3 root 6) over 3, by any valid route	M1	Method - expanding the numerator and the denominator	✓
Give a = 9 and b = 1	A1	Accuracy - final answer	✓

Full marks: 4/4

Question 12 - Exam Solution

Understanding the Question

Given

Ramp (hypotenuse) $= 10 + \sqrt{7}$

Find

Ground $= 5 + 2\sqrt{7}$

A right-angled triangle. The right angle is where the platform meets the ground.

The exact height h, in simplest surd form. Exact means no decimals, so the answer stays as a surd.

Plan the Solution

- The ramp is the hypotenuse, so Pythagoras gives $h^2 = (\text{ramp})^2 - (\text{ground})^2$.
- Square each bracket fully. Each one gives three terms before collecting.
- Watch what happens to the surd terms. They are the reason this question works.
- Take the square root at the very end, and simplify it.

Worked Solution [4 marks]

Rule - Pythagoras with surd sides: $h^2 = c^2 - b^2$, where c is the hypotenuse. Square each bracket in full: $(p + q\sqrt{n})^2 = p^2 + 2pq\sqrt{n} + q^2 n$. Do not take the square root until the very end.

Step 1: Write down Pythagoras.

$$h^2 = (10 + \sqrt{7})^2 - (5 + 2\sqrt{7})^2$$

(Reason: the ramp is the longest side, so it is the hypotenuse. The height is found by subtracting, not adding.)

Step 2: Square the hypotenuse.

$$(10 + \sqrt{7})^2 = 100 + 20\sqrt{7} + 7 = 107 + 20\sqrt{7}$$

(Reason: root seven squared is seven, which is where the whole number comes from.)

Step 3: Square the ground.

$$(5 + 2\sqrt{7})^2 = 25 + 20\sqrt{7} + 4 \times 7 = 53 + 20\sqrt{7}$$

(Reason: two root seven, squared, is four times seven, not two times seven. Squaring the two as well is the step most students miss.)

Step 4: Subtract, then take the square root.

$$h^2 = (107 + 20\sqrt{7}) - (53 + 20\sqrt{7}) = 54$$
$$h = \sqrt{54} = \sqrt{9 \times 6} = 3\sqrt{6}$$

(Reason: both squares produced twenty root seven, so the surds cancel exactly and h squared is a whole number. That is not luck. It is why these numbers were chosen.)

$$h = 3\sqrt{6}\ \text{m}$$

Verification

✓ **Check 1:** Pythagoras: $54 + 105.9150 = 159.9150$, and the ramp squared is 159.9150. The two agree exactly.

✓ **Check 2:** The ramp ≈ 12.6458 is longer than the ground ≈ 10.2915, as a hypotenuse must be, and $h \approx 7.3485 > 0$. The triangle is real.

✓ **Check 3:** $54 = 9 \times 6$, and $6 = 2 \times 3$ has no repeated prime, so six has no square factor. $3\sqrt{6}$ cannot be reduced any further.

Mark Scheme Breakdown

Step	Mark	Description	Got it?
Use Pythagoras: h squared = (10 + root 7) squared minus (5 + 2 root 7) squared	M1	Method - Pythagoras	✓
Expand both squares: 107 + 20 root 7, and 53 + 20 root 7	M1	Method - expanding both brackets	✓
Obtain h squared = 54	M1	Method - the surds cancel	✓
Give h = 3 root 6	A1	Accuracy - final answer	✓

Full marks: 4/4

Question 13 - Exam Solution

Understanding the Question

Given

$$\frac{7\sqrt{2}}{3+\sqrt{2}} + \frac{4+\sqrt{2}}{2-\sqrt{2}}$$

Two fractions, each with a two-term denominator containing a surd.

Find

The integers a and b in $a + b\sqrt{2}$. Multiplying by $\sqrt{2}$ alone will not clear either denominator. Each one needs its own conjugate.

Plan the Solution

- Two fractions cannot be added while a surd sits on the bottom of either one. Clear both denominators first.

- Each denominator has two terms, so each needs its own conjugate: the same two terms with the middle sign flipped.
- Rationalise each fraction separately, all the way to a simplified form.
- Only then add the two results, collecting the whole numbers and the surds separately.

Worked Solution [5 marks]

> **Rule - Rationalising a two-term denominator:** multiply the top and the bottom by the conjugate, which is the same expression with the middle sign flipped. Then $(c + d\sqrt{n})(c - d\sqrt{n}) = c^2 - d^2n$, a whole number, and the surd is gone. Two fractions can only be added once both denominators are whole numbers.

Step 1: Rationalise the first fraction.

$$\frac{7\sqrt{2}}{3+\sqrt{2}} \times \frac{3-\sqrt{2}}{3-\sqrt{2}} = \frac{7\sqrt{2}(3-\sqrt{2})}{(3+\sqrt{2})(3-\sqrt{2})}$$
$$= \frac{21\sqrt{2}-14}{9-2} = \frac{21\sqrt{2}-14}{7} = 3\sqrt{2}-2$$

(Reason: the conjugate of 3 + root 2 is 3 - root 2. Multiplying the top and the bottom by it is multiplying by one, so the value does not change. Only its form does. Root two times root two is two, and that is where the whole number comes from.)

Step 2: Rationalise the second fraction.

$$\frac{4+\sqrt{2}}{2-\sqrt{2}} \times \frac{2+\sqrt{2}}{2+\sqrt{2}} = \frac{(4+\sqrt{2})(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})}$$
$$= \frac{8+4\sqrt{2}+2\sqrt{2}+2}{4-2} = \frac{10+6\sqrt{2}}{2} = 5+3\sqrt{2}$$

(Reason: this denominator has a minus in the middle, so its conjugate has a plus. All four products on the top. Every term on the top is then divided by the whole number left on the bottom.)

Step 3: Add the two results.

$$(3\sqrt{2}-2)+(5+3\sqrt{2})$$
$$= (-2+5)+(3+3)\sqrt{2} = 3+6\sqrt{2}$$

(Reason: a whole number and a surd never merge. They are two separate running totals.)

$$a = 3, \quad b = 6$$

Verification

✓ **Check 1:** The expression evaluates to 11.4853 and $3+6\sqrt{2} \approx 11.4853$, so the two agree numerically.

✓ **Check 2:** Put the two fractions over a common denominator first, and rationalise only once: $\dfrac{7\sqrt{2}(2-\sqrt{2})+(4+\sqrt{2})(3+\sqrt{2})}{(3+\sqrt{2})(2-\sqrt{2})} = \dfrac{(14\sqrt{2}-14)+(14+7\sqrt{2})}{4-\sqrt{2}} = \dfrac{21\sqrt{2}}{4-\sqrt{2}}$, and then $\dfrac{21\sqrt{2}(4+\sqrt{2})}{14} = \dfrac{42+84\sqrt{2}}{14} = 3+6\sqrt{2}$. Same answer, a completely different route.

✓ **Check 3:** The second denominator $2-\sqrt{2}\approx 0.59$ is tiny, so that fraction alone is already about 9.24. Adding a positive first fraction must push the total above 9, and $3+6\sqrt{2}\approx 11.49$. An answer below 9 would be wrong on sight. Also, 2 is prime, so it carries no square factor: the form $a+b\sqrt{2}$ is already fully simplified.

Mark Scheme Breakdown

Step	Mark	Description	Got it?
Multiply the first fraction, top and bottom, by the conjugate 3 - root 2	**M1**	Method - conjugate of the first denominator	✓
Obtain 3 root 2 - 2	**A1**	Accuracy - first fraction	✓
Multiply the second fraction, top and bottom, by the conjugate 2 + root 2	**M1**	Method - conjugate of the second denominator	✓
Obtain 5 + 3 root 2	**A1**	Accuracy - second fraction	✓
Give a = 3 and b = 6	**A1**	Accuracy - final answer	✓

Full marks: 5/5

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